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Bilinear restriction theorems and applications to dispersive equations
(joint work with Pascal Bégout, Sahbi Keraani, Sanghyuk Lee, and Keith Rogers)

Abstract: The classical Strichartz's inequalities (Stein-Tomas theorem) assert that, given a solution $e^{it\Delta}u_0$, of the initial value problem

$$i\frac{\partial u}{\partial t} + \Delta u = 0,\ (t,x) \in \mathbf{R} \times \mathbf{R}^N,$$

$$u(0) = u_0,\ \text{in } \mathbf{R}^N,$$

then, for $q = 2\frac{N+2}{N}$, we have the apriori estimate

$$\|e^{it\Delta}u_0\|_{L^q(R\times R^N)} \leq C\|u_0\|_{L^2(R^N)}. \quad (1)$$

There is a similar theorem for the wave equation. These estimates have been used during the last thirty years to study nonlinear dispersive equations.

More recently, some new estimates, known as "bilinear" restriction theorems have been developed in the work of Bourgain, Wolff, Tao, Vega, Vargas, etc. The sharp version (in the L^2 case) is due to Wolff and Tao. Given two solutions of the initial value problem, $e^{it\Delta}u_0$, and $e^{it\Delta}v_0$, and under some assumptions on the support of $\widehat{u}_0$ and $\widehat{v}_0$,

$$\|e^{it\Delta}u_0 e^{it\Delta}v_0\|_{L^q(R\times R^N)} \leq C\|u_0\|_{L^2(R^N)}\|v_0\|_{L^2(R^N)},$$

for $q = \frac{N+3}{N+1}$.

These estimates can be used to improve the classical Strichartz estimate (**??**), replacing the L^2 norm in the right hand side by a smaller one, in a family of norms introduced by Bourgain in the 90's. This has some applications for the nonlinear L^2–critical Schrödinger equation

$$i\frac{\partial u}{\partial t} + \Delta u \pm |u|^{4/N}u = 0,\ (t,x) \in (-T_*, T_*) \times \mathbf{R}^N.$$

We can prove that for solutions with data in L^2 that blow up in finite time, there is a mass concentration phenomena. Moreover, we can prove the existence of solutions with minimal mass that blow up in finite time and some compactness results for the solutions of the equation. In dimension 2 this was proven by Bourgain, Merle-Vega and Keraani, using an improved Strichartz inequality due to Moyua-Vargas-Vega and in the 1 dimensional case by Carles-Keraani. For higher dimensions, it was proven via bilinear restriction estimates. Moreover, we can show that for data in some Sobolev space H^s, for some $s < 1$, the Duhamel part of the solution is more regular than the data. These results could be useful to prove global existence for data with low regularity, as in the work of Tao-Visan-Zhang (for radial data).

In the case of the wave equation, bilinear restriction estimates can be used to prove sharp "Null form estimates". Those were introduced in connection

with some nonlinear wave equations, related for instance to wave maps, and studied by several authors (Beals, Klainerman, Machedon, Selberg, Foschi, Tao, Tataru). This is also related to some other problems in Harmonic Analysis as the multiplier of the cone.

We will survey these results.