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*The sharp Sobolev inequality in quantitative form*

**Abstract:** For $n \geq 2$ and $1 < p < n$, let $p^* := np/(n-p)$. The classical *Sobolev inequality* states that there exists a positive constant $S(n,p)$ such that

$$S(n,p)\left(\int_{\mathbb{R}^n} |f|^{p^*}\right)^{1/p^*} \leq \left(\int_{\mathbb{R}^n} |\nabla f|^p\right)^{1/p}, \quad \forall f \in W^{1,p}(\mathbb{R}^n)\,. \tag{1}$$

The largest possible value for $S(n,p)$ is determined by the variational problem

$$S(n,p) = \inf\left\{ \frac{\left(\int_{\mathbb{R}^n} |\nabla f|^p\right)^{1/p}}{\left(\int_{\mathbb{R}^n} |f|^{p^*}\right)^{1/p^*}} : f \in X_p \right\}.$$

The infimum is achieved (i.e., equality holds in (1)) if and only if $f = g_{a,r,x_0}$ ([1], [6], [5]), where

$$g_{a,r,x_0}(x) = \frac{a}{(1 + r|x - x_0|^{p'})^{(n-p)/p}}\,, \quad a \neq 0\,, r > 0\,, x_0 \in \mathbb{R}^n\,. \tag{2}$$

Various extentions of this kind have been obtained in the literature, see for instance the paper [3] by H. Brézis and E. Lieb. A natural improvement of (1), conjectured in [3], is an inequality of the form

$$S(n,p)\|f\|_{p^*}\left(1 + \gamma_p(f)\right) \leq \|\nabla f\|_p, \tag{3}$$

where $\gamma_p$ is some non-negative function measuring the distance of $f$ from the manifold made up by all the optimal functions (2).
A positive answer to this question for the case $p = 2$ has been proved in [2] by G. Bianchi and H. Egnell. We present a result obtained in [4] which settles the problem for the general case $p \neq 2$. Namely we show that (3) holds with

$$\gamma_p(f) = \left( \inf_{a,r,x_0} \frac{\int_{\mathbb{R}^n} |f - g_{a,r,x_0}|^{p^*}}{\int_{\mathbb{R}^n} |f|^{p^*}} \right)^{\beta(n,p)},$$

for every $p \in (1,n)$, and for some exponent $\beta(n,p)$.

References

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